section 2.2

Linear Combinations

Motivation

Consider in $\mathbb{R}^3$ the coordinate vectors $\vec{i}=\begin{pmatrix}1\\0\\0\end{pmatrix}, \vec{j}=\begin{pmatrix}0\\1\\0\end{pmatrix}$ & $\vec{k}=\begin{pmatrix}0\\0\\1\end{pmatrix}$

Notice that any vector $v=\begin{pmatrix}a\\b\\c\end{pmatrix}$ in $\mathbb{R}^3$ can be obtained from these 3 vectors:

$$v=\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}a\\0\\0\end{pmatrix}+\begin{pmatrix}0\\b\\0\end{pmatrix}+\begin{pmatrix}0\\0\\c\end{pmatrix}=a\vec{i}+b\vec{j}+c\vec{k}.$$

example

$v=\begin{pmatrix}2\\3\\-1\end{pmatrix}=2\vec{i}+3\vec{j}-1\vec{k}$ (we say that v is obtained by adding scalar multiples of $\vec{i},\vec{j},\vec{k}$)

The vectors $\vec{i},\vec{j}$ & $\vec{k}$ are not unique we may find another set of vectors v_1, v_2, v_3 where v can be written as sum of scalar multiples of v_1, v_2 & v_3.

example $v_1=\begin{pmatrix}1\\1\\1\end{pmatrix}$ $v_2=\begin{pmatrix}1\\1\\0\end{pmatrix}$ $v_3=\begin{pmatrix}-1\\0\\0\end{pmatrix}$

$$v=\begin{pmatrix}a\\b\\c\end{pmatrix}=c\,v_1+(b-c)\,v_2+(b-a)\,v_3 \Rightarrow v=\begin{pmatrix}2\\3\\-1\end{pmatrix}=-v_1+4v_2+v_3$$

definition 2.2.1 (Linear Combination)

let $S=\{v_1, v_2, \dots, v_k\}$ be a set of vectors in $\mathbb{R}^n$ & let $c_1, c_2, \dots, c_k$ be scalars in $\mathbb{R}$. a linear Combination of the vectors of S is an expression of the form $c_1v_1+\dots+c_kv_k=\sum_{i=1}^{k} c_iv_i$

we say that a vector v is a linear Combination of the vectors in S if $v=c_1v_1+\dots+c_kv_k$ for some $c_1, c_2, \dots, c_k$ scalar in $\mathbb{R}$

($\Leftrightarrow \exists\, c_1, c_2, \dots, c_k \in \mathbb{R}$ such that $v=c_1v_1+\dots+c_kv_k$)

example

1. Determine whether $v=\begin{pmatrix}-1\\1\\10\end{pmatrix}$ is a linear Combination of $v_1=\begin{pmatrix}1\\0\\1\end{pmatrix}$, $v_2=\begin{pmatrix}-2\\3\\-2\end{pmatrix}$ & $v_3=\begin{pmatrix}-6\\7\\5\end{pmatrix}$

2. Determine whether the vector $v=\begin{pmatrix}1\\1\\1\end{pmatrix}$ is a linear Combination of $v_1=\begin{pmatrix}1\\0\\1\end{pmatrix}$, $v_2=\begin{pmatrix}1\\1\\0\end{pmatrix}$ & $v_3=\begin{pmatrix}2\\-1\\3\end{pmatrix}$

Answer

1 v lin Comb of $v_1, v_2, v_3 \iff \exists\, c_1, c_2, c_3$ scalars in $\mathbb{R}$ / $c_1v_1 + c_2v_2 + c_3v_3 = v$

$\iff \exists\, c_1, c_2, c_3$ scalar in $\mathbb{R}$ / $c_1\begin{pmatrix}1\\0\\1\end{pmatrix} + c_2\begin{pmatrix}-2\\3\\-2\end{pmatrix} + c_3\begin{pmatrix}-6\\7\\5\end{pmatrix} = \begin{pmatrix}-1\\1\\10\end{pmatrix}$

$\iff \exists\, c_1, c_2, c_3$ scalars in $\mathbb{R}$ / $\begin{pmatrix}c_1 - 2c_2 - 6c_3\\ 3c_2 + 7c_3\\ c_1 - 2c_2 + 5c_3\end{pmatrix} = \begin{pmatrix}-1\\1\\10\end{pmatrix}$

$\iff \exists\, c_1, c_2, c_3$ scalar in $\mathbb{R}$ / $\begin{cases} c_1 - 2c_2 - 6c_3 = -1\\ 3c_2 + 7c_3 = 1\\ c_1 - 2c_2 + 5c_3 = 10\end{cases}$

$\iff$ the system $\left(\begin{array}{ccc|c} 1 & -2 & -6 & -1\\ 0 & 3 & 7 & 1\\ 1 & -2 & 5 & 10\end{array}\right)$ is Consistent.

Study Consistency of the system

$\left(\begin{array}{ccc|c} 1 & -2 & -6 & -1\\ 0 & 3 & 7 & 1\\ 1 & -2 & 5 & 10\end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -2 & -6 & -1\\ 0 & 3 & 7 & 1\\ 0 & 0 & 11 & 11\end{array}\right) \Rightarrow$ the system is Consistent

$\iff v$ lin Combination of v_1, v_2, v_3

(not necessarily to solve the system)

to represent v as lin Comb of v_1, v_2, v_3 we need to find c_1, c_2 & c_3 & therefore solve the system $\Rightarrow \begin{cases} c_1 - 2c_2 - 6c_3 = -1\\ 3c_2 + 7c_3 = 1\\ 11c_3 = 11\end{cases} \Rightarrow \begin{array}{l} c_3 = 1\\ c_2 = -2\\ c_1 = -4 + 6 - 1 = 1\end{array} \Rightarrow v = v_1 - 2v_2 + v_3$

2 v linear Combination of $v_1, v_2, v_3 \iff \exists\, c_1, c_2, c_3$ scalars in $\mathbb{R}$ / $c_1v_1 + c_2v_2 + c_3v_3 = v$

$\iff \exists\, c_1, c_2, c_3$ scalars in $\mathbb{R}$ / $c_1\begin{pmatrix}1\\0\\1\end{pmatrix} + c_2\begin{pmatrix}1\\1\\0\end{pmatrix} + c_3\begin{pmatrix}2\\-1\\3\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$

$\iff \exists\, c_1, c_2, c_3$ scalars in $\mathbb{R}$ / $\begin{pmatrix}c_1 + c_2 + 2c_3\\ c_2 - c_3\\ c_1 + 3c_3\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$

$\iff \exists\, c_1, c_2, c_3$ scalars in $\mathbb{R}$ / $\begin{cases} c_1 + c_2 + 2c_3 = 1\\ c_2 - c_3 = 1\\ c_1 + 3c_3 = 1\end{cases}$

$\iff$ the system $\left(\begin{array}{ccc|c} 1 & 1 & 2 & 1\\ 0 & 1 & -1 & 1\\ 1 & 0 & 3 & 1\end{array}\right)$ is Consistent

Study Consistency

$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 1\\ 0 & 1 & -1 & 1\\ 1 & 0 & 3 & 1\end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1\\ 0 & 1 & -1 & 1\\ 0 & -1 & 1 & 0\end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1\\ 0 & 1 & -1 & 1\\ 0 & 0 & 0 & 1\end{array}\right)$ inconsistent

$\Rightarrow v$ is not a linear Combination of v_1, v_2 & v_3

(Notice here that $\boxed{v_3 = 3v_1 - v_2}$ v_3 lin Comb of v_1 & v_2)

example 2.

The Coordinate vectors in $\mathbb{R}^n$: $e_1=\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix}, e_2=\begin{pmatrix}0\\1\\0\\ \vdots\\0\end{pmatrix}, \dots, e_n=\begin{pmatrix}0\\0\\ \vdots\\1\end{pmatrix}$ $\left(e_i=\begin{pmatrix}0\\ \vdots\\0\\1\\0\\ \vdots\\0\end{pmatrix}\to i\text{th comp}\right)$

any vector $v=\begin{pmatrix}v_1\\ \vdots\\v_n\end{pmatrix}$ in $\mathbb{R}^n$ can be written as a linear Combination of $e_1,e_2,\dots,e_n \Rightarrow v=v_1e_1+\dots+v_ne_n$

example 3

show that the matrix $A=\begin{pmatrix}1&1\\1&0\end{pmatrix}$ is a linear Combination of $M_1=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $M_2=\begin{pmatrix}0&1\\1&1\end{pmatrix}$ & $M_3=\begin{pmatrix}1&1\\1&1\end{pmatrix}$

answer

A lin Comb of M_1, M_2 & $M_3 \Leftrightarrow \exists c_1, c_2$ & c_3 scalars in $\mathbb{R}$ / $c_1M_1+c_2M_2+c_3M_3=A$

$\Leftrightarrow \exists c_1,c_2,c_3$ scalars in $\mathbb{R}$ / $c_1\begin{pmatrix}1&0\\0&1\end{pmatrix}+c_2\begin{pmatrix}0&1\\1&1\end{pmatrix}+c_3\begin{pmatrix}1&1\\1&1\end{pmatrix}=\begin{pmatrix}1&1\\1&0\end{pmatrix}$

$\Leftrightarrow \exists c_1,c_2,c_3$ scalars in $\mathbb{R}$ / $\begin{pmatrix}c_1+c_3 & c_2+c_3\\ c_2+c_3 & c_1+c_2+c_3\end{pmatrix}=\begin{pmatrix}1&1\\1&0\end{pmatrix}$

$\Leftrightarrow \exists c_1,c_2,c_3$ scalars in $\mathbb{R}$ / $\begin{cases}c_1+c_3=1\\c_2+c_3=1\\c_2+c_3=1\\c_1+c_2+c_3=0\end{cases}$

$\Leftrightarrow$ the system $\left(\begin{array}{ccc|c}1&0&1&1\\0&1&1&1\\0&1&1&1\\1&1&1&0\end{array}\right)$ is Consistent

study Consistency

$\left(\begin{array}{ccc|c}1&0&1&1\\0&1&1&1\\0&1&1&1\\1&1&1&0\end{array}\right)\to\left(\begin{array}{ccc|c}1&0&1&1\\0&1&1&1\\0&0&0&0\\0&1&0&-1\end{array}\right)\to\left(\begin{array}{ccc|c}1&0&1&1\\0&1&0&-1\\0&1&1&1\\0&0&0&0\end{array}\right)\to\left(\begin{array}{ccc|c}1&0&1&1\\0&1&0&-1\\0&0&1&2\\0&0&0&0\end{array}\right)$ Consistent

& $c_3=2 \quad c_2=-1 \quad c_1=-1$

$\Rightarrow A=-M_1-M_2+2M_3$

example 4

if $y_1,y_2,\dots,y_k$ are k solutions of the homogenous linear system $AX=0$ show that any linear Combination of $y_1,y_2,\dots,y_k$ is also a solution of $AX=0$

Answer

let y be a lin Combination of $y_1,y_2,\dots,y_k \Leftrightarrow \exists c_1,c_2,\dots,c_k\in\mathbb{R}$ / $y=c_1y_1+\dots+c_ky_k$.

show that y is a solution of $AX=0$ (ie show that $Ay\overset{??}{=}0$)

$Ay=A(c_1y_1+\dots+c_ky_k)\overset{\text{distr}}{=}A(c_1y_1)+\dots+A(c_ky_k)=c_1(Ay_1)+\dots+c_k(Ay_k)$ (properties of matrices)

$=c_1(0)+\dots+c_k(0)$ (since $Ay_i=0\ \forall i=1,\dots,k$; y_i sol of $AX=0$)

$=0$

$\Rightarrow y=c_1y_1+\dots+c_ky_k$ solution of $AX=0$

Important result

If $y_1, y_2, \ldots, y_k$ are solutions of $AX=0$ (homogenous linear system) then any linear Combination of $y_1, y_2, \ldots, y_k$ is also a solution of $AX=0$

But this is not true for non homogenous linear systems $AX=b\neq 0$; if $y_1, y_2, \ldots, y_k$ are solutions of $AX=b\neq 0$

$\not\Rightarrow$ $y=c_1y_1+c_2y_2+\cdots+c_ky_k$ solution of $AX=b$

(to disprove take any numerical counter example)

Remark

Representation of the matrix multiplication AX with respect to the row vectors of A & with respect to the column vectors of A (A is any $m\times n$ matrix & X any $n\times 1$ vector)

(a) representation of AX with respect to the row vectors $R_1, R_2, \ldots, R_m$ of A

$$A\cdot X = \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{pmatrix} \cdot X = \begin{pmatrix} R_1\cdot X \\ R_2\cdot X \\ \vdots \\ R_m\cdot X \end{pmatrix}$$

(b) representation of AX with respect to the column vectors $A_1, A_2, \ldots, A_n$ of A (very important)

$$AX = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n \\ a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n \\ \vdots \\ a_{m1}x_1+a_{m2}x_2+\cdots+a_{mn}x_n \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}x_1 \\ a_{21}x_1 \\ \vdots \\ a_{m1}x_1 \end{pmatrix} + \begin{pmatrix} a_{12}x_2 \\ a_{22}x_2 \\ \vdots \\ a_{m2}x_2 \end{pmatrix} + \cdots + \begin{pmatrix} a_{1n}x_n \\ a_{2n}x_n \\ \vdots \\ a_{mn}x_n \end{pmatrix} = \underbrace{\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}}_{A_1} x_1 + \underbrace{\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}}_{A_2} x_2 + \cdots + \underbrace{\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}}_{A_n} x_n$$

$\Rightarrow AX = x_1A_1 + x_2A_2 + \cdots + x_nA_n$

Therefore AX is a linear Combination of the column vectors of A

Theorem 2.21.

Consider the $m \times n$ linear system $AX = b$

$AX = b$ is Consistent $\Longleftrightarrow$ the vector b can be expressed as a linear Combination of the column vectors of A

proof

given $AX = b$ Consistent $\Longleftrightarrow$ $AX = b$ has at least one solution

$\Longleftrightarrow \exists$ at least one $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ solut of $AX = b$

by previous RemarK any AX is a linear Combinat of the column vectors of A ie $AX = x_1 A_1 + x_2 A_2 + \cdots + x_n A_n$.

$\Longleftrightarrow \exists\, x_1, x_2, \ldots, x_n \in \mathbb{R} \,/\, x_1 A_1 + x_2 A_2 + \cdots + x_n A_n = b$

$\Longleftrightarrow$ b is a linear Combination of the column vectors $A_1, A_2, \ldots, A_n$ of A.